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Generalization of Nambu's Mechanics

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Abstract

We look for a generalization of the mechanics of Hamilton and Nambu. We have found the equations of motion of a classical physical system of S basic dynamic variables characterized by $S - 1$ constants of motion and by a function of the dynamical variables and the time whose value also remains constant during the evolution of the system. The number S may be even or odd. We find that any locally invertible transformations are canonical transformations. We show that the equations of motion obtained can be put in a form similar to Nambu's equations by means of a time transformation. We study the relationship of the present formalism to Hamiltonian mechanics and consider an extension of the formalism to field theory.

1. *Introduction*

In the formalism of Hamiltonian mechanics a physical system is characterized by N canonical pairs of coordinates and momenta $q_1, p_1, \ldots q_N, p_N$, and by a Hamiltonian $H(q, p)$. The equations of motion are obtained with the help of a variational principle.

Recently, Nambu (1973) considered a generalization of Hamiltonian mechanics to the case of a three-dimensional phase space intead of the conventional phase space spanned by a canonical pair (q, p). He postulates, given a triplet (x, y, z) of dynamical variables and a pair of 'Hamiltonians' $H(x, y, z)$, $G(x, y, z)$, the following equations:

$$\frac{dx}{dt} = \frac{\partial(H, G)}{\partial(y, z)}, \qquad \frac{dy}{dt} = \frac{\partial(H, G)}{\partial(z, x)}, \qquad \frac{dz}{dt} = \frac{\partial(H, G)}{\partial(x, y)} \tag{1.1}$$

For any $F(x, y, z)$, then, we have

$$\frac{dF}{dt} = \frac{\partial(F, H, G)}{\partial(x, y, z)} \equiv [F, H, G] \tag{1.2}$$

Nambu called $[F, H, G]$ the generalized Poisson bracket of this theory.

For the case of a phase space of any dimensionality, n, he introduces $n-1$ Hamiltonians H_k, and postulates:

$$\frac{dF}{dt} = \frac{\delta(F, H_1, H_2, \ldots, H_{n-1})}{\delta(x_1, x_2, \ldots, x_n)} \tag{1.3}$$

in lieu of (1.1).

In that paper he restricts himself to the three-dimensional case, and justifies the physical relevance of such generalizations because equations (1.1) is nothing but the Euler equation for a rigid rotator, if we identify x, y, z with the angular momentum components Lx, Ly, Lz in the body fixed frame on the one hand, and G and H, respectively, with the total kinetic energy and the square of the angular momentum in that frame,

$$G = \frac{1}{2}\left(\frac{L_x{}^2}{I_x} + \frac{L_y{}^2}{I_y} + \frac{L_z{}^2}{I_z}\right), \qquad H = \tfrac{1}{2}(L_x{}^2 + L_y{}^2 + L_z{}^2) \tag{1.4}$$

The main purpose of the present work is to study a physical system of S basic dynamical variables $x_1, x_2 \ldots, x_S$ characterized by $S-1$ constants of motion $G_r(x)$ and by a function $W(x, t)$ whose value also remains constant during the evolution of the system.

We find that the equations of motion are unique and that they can be given the form of the equations (1.3) postulated by Nambu by means of a time transformation.

In the analysis of invariant properties of the equations obtained we find that the set of canonical transformations is formed by any locally invertible transformation of the dynamical variables. On the other hand Nambu's equations are invariant under a more restricted group.

It is worth pointing out that the formalism developed is the most general consistent with the existence of the G's and the W with the additional assumption of an invertibility hypothesis.†

The plan of the work is as follows: In Section 2 the formalism is elaborated. The relation between the equations obtained in Section 2 and Nambu's equations are studied in Section 3, and in Section 4 we analyze the relationship with Hamiltonian mechanics. Section 5 refers to the invariant properties, and in Section 6 the formalism is applied to the study of the free particle and the harmonic oscillator. Finally, in Section 7 we consider a possible generalization of the formalism to field theory and it is applied to the study of the Schrödinger field.

† The present work is closely related to the theory of Jacobi's multipliers (Apell, 1953, pp. 465–474; de La Vallée Poussin, 1949, pp. 306–313; Pars, 1968, pp. 409–413; Kilmister, 1964, pp. 80–85) with which some of the mathematical expressions coincide, but our theory differs from Jacobi's since both the objectives and logical sequence used are different.

2. *Formalism*

We want to describe a classical physical system in which the state is fully characterized by S independent variables $\{x_s\}$ which evolve in time. The number S can be even or odd.

We consider the non-trivial case in which the physical system is characterized by $S - 1$ constants of motion, say

$$G_r = G_r(x), \qquad r = 1, 2, \ldots, S-1 \tag{2.1}$$

and by a function W,

$$W = W(x, t) \tag{2.2}$$

whose value also remains constant during the evolution of the system. Moreover the G's and W are assumed to be independent functions with respect to the variables x.

The fact that the G's are constants of motion means that

$$\frac{d}{dt}\{G_r[x(t, x^0)]\} = 0, \qquad r = 1, 2, \ldots, S-1 \tag{2.3}$$

for any initial state x^0.

These equations are equivalent to

$$\sum_{s=1}^{S} \frac{\partial G_r}{\partial x_s}\frac{d x_s}{dt} = 0, \qquad r = 1, 2, \ldots, S-1 \tag{2.4}$$

We also have

$$\frac{d}{dt}\{W[x(t, x^0), t]\} = 0 \tag{2.5}$$

or

$$\sum_{s=1}^{S} \frac{\partial W}{\partial x_s}\frac{dx_s}{dt} + \frac{\partial W}{\partial t} = 0 \tag{2.6}$$

The relations (2.4) and (2.6) form a linear system of S equations in the S unknowns dx_s/dt. The determinant formed by the coefficients of the velocities in the relations (2.4) and (2.6) is different from zero since the G's and W are functionally independent.

Using the properties of the determinants and the independence of the x's we can set the unique solution in the following form:

$$\frac{dx_s}{dt} = \left(-\frac{\partial W}{\partial t}\right) \frac{\dfrac{\partial(x_s, G_1, G_2, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}}{\dfrac{\partial(W, G_1, G_2, \ldots, x_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}}, \qquad s = 1, 2, \ldots S \tag{2.7}$$

For any function $F(x)$, we have

$$\frac{dF}{dt} = \left(-\frac{\partial W}{\partial t}\right) \frac{\dfrac{\partial(F, G_1, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}}{\dfrac{\partial(W, G_1, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}} \tag{2.8}$$

Equations (2.7) or (2.8) are the equations of motion of a physical system characterized by the functions (2.1) and (2.2).

We may define a generalized Poisson bracket $\begin{Bmatrix} F, G_1, G_2, \ldots, G_{S-1} \\ W, G_1, G_2, \ldots, G_{S-1} \end{Bmatrix}$ as

$$\begin{Bmatrix} F, G_1, G_2, \ldots, G_{S-1} \\ W, G_1, G_2, \ldots, G_{S-1} \end{Bmatrix} \equiv \frac{\dfrac{\partial(F, G_1, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}}{\dfrac{\partial(W, G_1, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}} \tag{2.9}$$

Hence we can write (2.8) as

$$\frac{dF}{dt} = \left(-\frac{\partial W}{\partial t}\right) \begin{Bmatrix} F, G_1, \ldots, G_{S-1} \\ W, G_1, \ldots, G_{S-1} \end{Bmatrix} \tag{2.10}$$

Note that if F is equal to any of the G's the bracket (2.9) is zero bringing out its character as constants of motion. Moreover the bracket (2.9) is symmetric under interchange of any pair of the G's revealing their equal status. Finally it is easy to see that the bracket (2.9) is linear in F and obeys the derivation law.

3. *Relation With Nambu's Equations*

The equation of motion (2.8) can be put in a form similar to Nambu's equation by means of a time transformation. Consider

$$t' \equiv \int_{t_0}^{t} \left(-\frac{\partial W}{\partial \tau}\right) \frac{1}{\dfrac{\partial(W, G_1, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}} \, d\tau \tag{3.1}$$

then

$$\frac{dt'}{dt} = \left(-\frac{\partial W}{\partial t}\right) \frac{1}{\dfrac{\partial(W, G_1, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}} \tag{3.2}$$

Note that the right-hand side of (3.2) is always different from zero and this implies that t' is a monotonous function of t.

Using (3.2) we obtain that (2.8) can be written as

$$\frac{dF}{dt'} = \frac{\partial(F, G_1, G_2, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)} \tag{3.3}$$

4. *Relation to Hamiltonian Mechanics*

A closed mechanical system of N coordinates q_n and N canonically conjugated momenta p_n can be characterized by a Hamiltonian H, and it has as equations of motion

$$\dot{q}_n = \frac{\partial H}{\partial p_n}, \qquad \dot{p}_n = -\frac{\partial H}{\partial q_n}, \qquad n = 1, 2, \ldots, N \tag{4.1}$$

These equations form a system of $2N$ differential equations of first order. Then the general solution has $2N$ arbitrary constants. To fix them it is necessary to know the state of the system at some instant. The Hamiltonian does not contain explicitly the time (closed mechanical system), and this implies that Hamilton's equations do not contain t explicitly. Hence the election of the time's origin is completely arbitrary and one of the constants in the solution of equations (4.1) may be considered as an additive constant t_0 to the time.†

The general solution of the equations of motion will be

$$\begin{aligned} q_n &= q_n(t + t_0, C_1, \ldots, C_{2N-1}), \qquad n = 1, 2, \ldots, N \\ p_n &= p_n(t + t_0, C_1, \ldots, C_{2N-1}), \qquad n = 1, 2, \ldots, N \end{aligned} \tag{4.2}$$

Formally, we may solve for t_0 from equations (4.2),

$$t_0 = f(q, p, C_1, \ldots, C_{2N-1}) - t \tag{4.3}$$

and substituting equation (4.3) in (4.2) we can express the $2N - 1$ arbitrary constants as functions of the q's and p's:

$$C_n = C_n(q, p), \qquad n = 1, 2, \ldots, 2N - 1 \tag{4.4}$$

The functions (4.4) are constants of motion of the system.† On the other hand we can make use of (4.4) to express $t + t_0$ as a function of the q's and p's only. Hence in this problem the G's of Section 2 will be given by the C's and W will be given by (4.3). If we apply the formalism of Section 2 we infer equations like (2.8).

5. *Invariance Properties*

5.1. *Canonical Transformations*

In the formalism of Hamiltonian mechanics canonical transformations are defined as those transformations of the dynamical variables that leave invariant

† See e.g. Landau & Lifshitz (1969, p. 23).

the Hamiltonian formalism. It is proved that the canonical transformations leave invariant the value of the fundamental Poisson brackets (Goldstein, 1964, p. 253).†

Taking into account this property of the canonical transformations in the Hamiltonian formalism, Nambu considers a transformation $(x, y, z) \to (x', y', z')$ as canonical *iff* it leaves invariant the fundamental (Nambu) bracket:

$$[x', y', z'] = [x, y, z] = 1 \qquad (5.1.1)$$

The form of Nambu's equations remains unaffected under a canonical transformation.

Similarly we define a transformation

$$(x_1, x_2, \ldots, x_S) \to (x'_1, x'_2, \ldots, x'_S) \qquad (5.1.2)$$

as canonical *iff* it leaves invariant the fundamental generalized bracket of the theory:

$$\begin{Bmatrix} x'_1, x'_2, \ldots, x'_S \\ x'_1, x'_2, \ldots, x'_S \end{Bmatrix} = 1 \qquad (5.1.3)$$

Clearly every locally invertible transformation, namely, a transformation such that

$$\frac{\partial(x'_1, x'_2, \ldots, x'_S)}{\partial(x_1, x_2, \ldots, x_S)} \neq 0 \qquad (5.1.4)$$

is canonical. Then

$$\frac{dF}{dt} = \left(-\frac{\partial W}{\partial t}\right) \frac{\dfrac{\partial(F, G_1, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}}{\dfrac{\partial(W, G_1, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}} = \left(-\frac{\partial W}{\partial t}\right) \frac{\dfrac{\partial(W, G_1, \ldots, G_{S-1})}{\partial(x'_1, x'_2, \ldots, x'_S)}}{\dfrac{\partial(F, G_1, \ldots, G_{S-1})}{\partial(x'_1, x'_2, \ldots, x'_S)}} \qquad (5.1.5)$$

That is, equations (2.8) are form invariant if we use the new set of canonical variables.

† This is also a sufficient condition (Goldstein, 1964, p. 272).

5.2. *Invariance Under a Transformation of the Constants of Motion*

We now study what happens to the equations of motion (2.8) when we change the original set of constants of motion $\{G_r\}$ by another set $\{H_r\}$ such that

$$\frac{\partial(H_1, H_2, \ldots, H_{S-1})}{\partial(G_1, G_2, \ldots, G_{S-1})} \neq 0 \tag{5.2.1}$$

If we use the following identity

$$\frac{\partial(F, H_1, H_2, \ldots, H_{S-1})}{\partial(F, G_1, G_2, \ldots, G_{S-1})} = \frac{\partial(W, H_1, H_2, \ldots, H_{S-1})}{\partial(W, G_1, G_2, \ldots, G_{S-1})} \tag{5.2.2}$$

we can write

$$\frac{dF}{dt} = \left(-\frac{\partial W}{\partial t}\right) \frac{\dfrac{\partial(F, G_1, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}}{\dfrac{\partial(W, G_1, \ldots, G_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}} = \left(-\frac{\partial W}{\partial t}\right) \frac{\dfrac{\partial(W, H_1, \ldots, H_{S-1})}{\partial(x_1, x_2, \ldots, x_S)}}{\dfrac{(F, H_1, \ldots, H_{S-1})}{(x_1, x_2, \ldots, x_S)}} \tag{5.2.3}$$

Hence substituting the original set of constants of motion by a new set of independent functions the physical system remains unaltered.

6. *Examples*

6.1. *Free Particle*

We take as dynamical variables the coordinates of the particle x_1, x_2, x_3, and their corresponding momenta x_4, x_5, x_6.

The constants of motion that characterize the system are

$$G_1 = x_4, \qquad G_2 = x_5, \qquad G_3 = x_6$$

$$G_4 = x_1 x_5 - x_4 x_2, \qquad G_5 = x_2 x_6 - x_3 x_5 \tag{6.1.1}$$

It is clear that they correspond to the linear momentum of the particle in the three spatial directions and to two of the components of the angular momenta.

The function W will be

$$W = x_1 - x_4 t \tag{6.1.2}$$

If we use equations (2.8) we are led to the usual equations of the free particle.

6.2. *Harmonic Oscillator*

We choose the variable x_1 proportional to the particle's position and the variable x_2 to the velocity, in such a way that the energy of the system is

$$G = \frac{x_1^2}{2} + \frac{x_2^2}{2} \tag{6.2.1}$$

We take

$$W = \cos^{-1} \frac{x_1}{\sqrt{x_1^2 + x_2^2}} + t \tag{6.2.2}$$

where $\cos^{-1}(x_1/\sqrt{x_1^2 + x_2^2})$ is an angle of the first or second quadrant. Note that W is nothing more than t_0 as a function of the dynamical variables and the time.

Using (2.8) we find

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 \tag{6.2.3}$$

namely, the usual equations of harmonic oscillator.

7. *An Extension of the Formalism to Classical Field Theory*

Until now we have considered only systems in which the number of basic dynamic variables is finite. In the case of a field the dynamical variables depend on a continuous index besides a discrete one. We write the fields variables as

$$\psi_{sx}, \qquad s = 1, 2, \ldots, S \tag{7.1}$$

with

$$x = (x_1, x_2, x_3)$$

We characterize the system with $S - 1$ functions G_r,

$$G_r = \int g_r(\psi, \nabla\psi)\, d^3x \tag{7.2}$$

constants of motion of the system, and by a function W

$$W = \int w(\psi, \nabla\psi, t)\, d^3x \tag{7.3}$$

whose value does not change in time, in such a way that the G's and W will be assumed to be functionally independent with respect to $\psi_{1x}, \psi_{2x}, \ldots, \psi_{Sx}$.

The fact that the G's are constants of motion means

$$\frac{dG_r}{dt} = \int \sum_{s=1}^{S} \frac{\delta G_r}{\delta \psi_{sx}} \frac{\partial \psi_{sx}}{\partial t}\, d^3x = 0 \tag{7.4}$$

and a sufficient condition to satisfy (7.4) is

$$\sum_{s=1}^{S} \frac{\delta G_r}{\delta \psi_{sx}} \frac{\partial \psi_{sx}}{\partial t} = 0 \tag{7.5}$$

Also we have

$$\sum_{s=1}^{S} \frac{\delta W}{\delta \psi_{sx}} \frac{\partial \psi_{sx}}{\partial t} + \frac{\partial w}{\partial t} = 0 \tag{7.6}$$

The relations (7.5) and (7.6) are a linear system of equations of the same kind as the one studied in Section 2. If we follow a similar analysis we finally find that

$$\frac{\partial\psi_{sx}}{\partial t} = \int\left(-\frac{\partial w}{\partial t}\right)\frac{\dfrac{\partial(\psi_{sx'}, G_1, \ldots, G_{S-1})}{\partial(\psi_{1x'}, \psi_{2x'}, \ldots, \psi_{Sx'})}}{\dfrac{\partial(W, G_1, \ldots, G_{S-1})}{\partial(\psi_{1x'}, \psi_{2x'}, \ldots, \psi_{1x'})}}\, d^3x' \tag{7.7}$$

In the same way as in Section 3 we may perform a time transformation that change equations (7.7) to

$$\frac{\partial\psi_{sx}}{\partial t} = \int \frac{\partial(\psi_{sx'}, G_1, \ldots, G_S)}{\partial(\psi_{1x'}, \psi_{2x'}, \ldots, \psi_{Sx'})}\, d^3x' \tag{7.8}$$

As an example of the application of the formalism† we consider the Schrödinger field (Schiff, 1955, p. 348). The dynamical variables are ψ_x, π_x, ψ_x^*, π_x^*. The constants of motion are

$$\begin{aligned} G_1 &= \int i\left[\frac{1}{2m}(\nabla\psi^* . \nabla\psi) + V\psi^*\psi\right] d^3x \\ G_2 &= \int (\pi_x - i\psi_x^*)\, d^3x \\ G_3 &= \int (\pi_x^* + i\psi_x)\, d^3x \end{aligned} \tag{7.9}$$

Using equations (7.8) we find that

$$-\frac{1}{2m}\nabla^2\psi_x + V\psi_x = i\frac{\partial\psi_x}{\partial t} \tag{7.10}$$

$$\frac{\partial}{\partial t}(\pi_x - i\psi_x^*) = 0 \tag{7.11}$$

and their complex conjugates.

Equation (7.10) is the usual Schrödinger equation. Equation (7.11) underlies the indetermination of the Lagrangian of a standard formalism under the addition of a four-divergence.

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† We emphasize that the conditions (7.5) and (7.6) only are sufficient conditions in order to note that equations (7.7) or (7.8) are not a unique consequence of the initial assumptions.

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